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# Sub-equation Method for the Conformable Fractional Generalized Kuramoto-Sivashinsky Equation

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**Keywords**

Travelling wave solutions, Sub-equation method, Generalized Kuramoto-Sivashinsky equation.

**Abstract**

In this paper, we find travelling wave solutions for the conformable fractional generalized Kuramoto-Sivashinsky equation. This equation arises in several problems of physics and chemistry.The sub-equationmethod is used to construct the travelling wave solutions of the conformable fractional generalized Kuramoto-Sivashinsky equation. As a result, the solutions obtained here are expressed in hyperbolic functions and trigonometric functions. Compared with other methods, this method is direct, concise, effective and easy to calculate, and it is a powerful mathematical tool for obtaining exact travelling wave solutions of other nonlinear conformable fractional partial differential equations.

## 1. Introduction

Most phenomena in science and engineering are expressed by partial differential equations (PDE). Fractional calculus work on the powers of the differential equations that are not integer. Fractional calculus was first developed as a pure mathematical theory in the middle of the 19th century [1]. The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [2].

Modeling of the physical phenomena always has been of interest to the scientists and fractional differential equation is a tool to modeling natural phenomena. Recently fractional differential equations (FDEs) has been more attention to the researchers because of its numerous applications in physics, chemistry, engineering, and even finance and social sciences [3,4].

For better understanding the mechanisms of the complicated nonlinear physical phenomena as well as further applying them in practical life, the solution of FDE is much involved. There are many methods to find exact or numerical solutions of nonlinear fractional differential equations (NFDEs). Among these methods we can name spline collocation method [5], predictor corrector method [6] as numerical methods, the first integral method [7], thetanh-function method [8], the as analytical methods. In this paper, we will implement the sub-equation method to obtain the exact solutions for the fractional generalized Kuramoto-Sivashinsky equation.

Many definitions for fractional differential equations exist like Grunwald-Letnikov, Riemann-Liouville, Caputo and Riesz fractional derivatives [2]. A new modification of Riemann-Liouville derivative is define by Jumarie [9]

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-\varepsilon)^{-\alpha}\big(f(\varepsilon)-f(0)\big)d\varepsilon, \qquad (1)$$

that $0<\alpha<1$. Some basic fractional calculus formulae given below [9]

$$D_x^\alpha\big(u(x)v(x)\big) = v(x)D_x^\alpha\big(u(x)\big) + u(x)D_x^\alpha\big(v(x)\big), \qquad (2)$$

$$D_x^\alpha\Big(f\big(u(x)\big)\Big) = f_u'(u)D_x^\alpha\big(u(x)\big) = D_x^\alpha f(u)(u_x')^\alpha. \qquad (3)$$

Eq. (3) has been applied to finding the exact solutions of some nonlinear fractional order differential equations. Recently, Liu [10] has shown that Jumarie's basic formulae (2) and (3) are not correct, and consequently the corresponding results on fractional differential equations are flawed. To overcome these and the related difficulties, Khalil et al. [11] introduced a new simple well-behaved definition of the fractional derivative called conformable fractional derivative. This fractional derivative is theoretically very easier to handle and also obeys some conventional properties that cannot be satisfied by the existing fractional derivatives, for instance, the chain rule [12]. However, this fractional derivative has a weakness, which is the fractional derivative of any differentiable function at the point zero. In short time,

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many studies related to this new fractional derivative definition was done [13,14].

Modified conformable fractional derivative is given as [11]

$$ {}_tT_\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}-f(t)\right)}{\varepsilon}, \quad (4)$$

in which $0<\alpha\le 1, f:(0,\infty)\to\mathbb{R}$ and $t>0$. Some properties for this modified conformable fractional derivative are as [11]

$$ {}_tT_\alpha\left(au(t)+bv(t)\right)=a\,{}_tT_\alpha u(t)+b\,{}_tT_\alpha v(t), a,b\in\mathbb{R}, \quad (5)$$

$$ {}_tT_\alpha t^\gamma=\gamma t^{\gamma-\alpha}, \qquad \gamma\in\mathbb{R}, \quad (6)$$

$$ {}_tT_\alpha\left(u(t)v(t)\right)=v(t)\,{}_tT_\alpha u(t)+u(t)\,{}_tT_\alpha v(t), \quad (7)$$

$$ {}_tT_\alpha\left(\frac{u(t)}{v(t)}\right)=\frac{v(t)\,{}_tT_\alpha u(t)-u(t)\,{}_tT_\alpha v(t)}{v^2(t)} \quad (8)$$

**Theorem 1. (Chain Rule [11])** Assume $f,g:(0,\infty)\to\mathbb{R}$ be $\alpha$-differentiable functions, where $0<\alpha\le 1$. Let $h(t)=f(g(t))$. Then $h(t)$ is $\alpha$-differentiable and for all $t$ with $t\neq 0$ and $g(t)\neq 0$ we have

$$({}_tT_\alpha h)(t)=({}_tT_\alpha f)\left(g(t)\right)({}_tT_\alpha g)(t)g(t)^{\alpha-1}.$$

The rest of this paper is organized as follows: In Section 2, we will describe the sub-equation method for solving conformable fractional differential equations (CFPDEs). In Section 3, we apply this method to establish exact solution for the conformable fractional generalized Kuramoto-Sivashinsky equation. Some conclusions are presented at the end of the paper.

## 2. The Sub-equation Method

Now, we describe the main steps of the sub-equation method for finding exact solutions of nonlinear CFPDEs.

***Step 1.*** Suppose that a nonlinear CFPDEs, say in the independent variables $x,y,t$ is given by

$$F\left(u,u_t,u_x,u_y,{}_tT_\alpha u,{}_xT_\alpha u,{}_yT_\alpha u,{}_tT_{2\alpha}u,\dots\right)=0, \quad (9)$$

where $0<\alpha\le 1$, ${}_tT_\alpha u$ , ${}_xT_\alpha u$ and ${}_yT_\alpha u$ are conformable fractional derivatives of $u$ with respect to $x,y$ and $t$ respectively, $u(x,y,t)$ is an unknown function, $F$ is a polynomial in $u$ and its various partial derivatives including conformable fractional derivatives.

***Step 2.*** By means of the traveling wave transformation

$$u(x,y,t)=u(\xi), \qquad \xi=k\frac{x^\alpha}{\alpha}+l\frac{y^\alpha}{\alpha}+c\frac{t^\alpha}{\alpha}, \quad (10)$$

where $k,l$ and $c$ are constants, Eq. (9) becomes the following nonlinear ordinary differential equation (ODE) with respect to the variable $\xi$

$$F(u,ku',lu',cu',ku'',lu'',cu'',\dots)=0. \quad (11)$$

***Step 3.*** We suppose that Eq. (11) has the following solution

$$u(\xi)=\sum_{i=0}^{n}a_i\,\varphi^i(\xi), \quad (12)$$

where $a_i(i=0,1,\dots,n)$ are constants to be determined later with $a_n\neq 0$. The positive integer $n$ can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (11) and $\varphi=\varphi(\xi)$ satisfies the following fractional Riccati equation

$$\varphi'=\sigma+\varphi^2, \quad (13)$$

where $\sigma$ is a constant. We know that Eq. (13) admit the following solutions

$$\varphi=\begin{cases}-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\,\xi) & \sigma<0,\\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\,\xi) & \sigma<0,\\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\,\xi\right) & \sigma>0,\\ -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\,\xi\right) & \sigma>0,\\ -\dfrac{1}{\xi+\omega},\quad \omega=const, & \sigma=0.\end{cases} \quad (14)$$

***Step 4.*** Substituting Eqs. (12) and (13) into Eq. (11) and collecting the coefficients of $\varphi(\xi)$ and setting the coefficients of $[\varphi(\xi)]^i(i=0,1,\dots)$ to be zero, we get an over-determined system of algebraic equations with respect to $a_i\,(i=0,1,\dots,n)$ and $k,l,c$.

***Step 5.*** Finally, assuming that $k,l,c$ and $a_i(i=0,1,\dots,n)$ are obtained by solving the algebraic equations in the previous step, and substituting these constants and the solutions of Eq. (12) into Eq. (11), we can obtain the explicit solutions of Eq. (9).

## 3. Application on the conformable fractional generalized Kuramoto-Sivashinsky equation

In this section, the sub-equation method is employed to construct the traveling wave solutions of the conformable fractional generalized Kuramoto-Sivashinsky equation.

Let us consider the conformable fractional generalized Kuramoto-Sivashinsky equation

$$ {}_tT_\alpha u+u\,{}_xT_\alpha u+\mu\,{}_xT_{2\alpha}u+\nu\,{}_xT_{3\alpha} +\eta\,{}_xT_{4\alpha}u=0, \quad t>0, \quad 0<\alpha\le 1, \quad (15)$$

which is the variation of the generalized Kuramoto-Sivashinsky equation [15]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\nu\frac{\partial^3 u}{\partial x^3}+\eta\frac{\partial^4 u}{\partial x^4}=0, \quad (16)$$

where $\mu,\nu$ and $\eta\neq 0$ are real constants, and $t>0$. This equation is also called KdV-Burgers-Kuramoto equation [16]. The generalized Kuramoto-Sivashinsky equation is a nonlinear evolution equation and has many applications in a variety of physical phenomena such as flame front instability, dissipative structure of reaction-diffusion, long waves on the interface between two viscous fluids, thin hydrodynamics films and unstable drift waves in plasmas. In generalized Kuramoto-Sivashinsky equations the linear terms counterbalance the nonlinear term, providing a mechanism for energy transfer between wave modes. At $\nu = 0$, Eq. (16) is reduced to as the Kuramoto-Sivashinsky equation, one of the simplest equations that appear in modeling the nonlinear behavior of disturbances for a sufficiently large class of active dissipative media. In the past several decades, various methods have been proposed to solve the generalized Kuramoto-Sivashinsky equation. Some of these methods are meshless method useing radial basis functions [17], Chebyshev spectral collocation methods [18], He's variational iteration method [19].

Now, we employ the sub-equation method to obtain the travelling wave solutions of the conformable fractional generalized Kuramoto-Sivashinsky equation.

Using the transformation

$$u(x,t) = u(\xi), \quad \xi = k\frac{x^\alpha}{\alpha} + c\frac{t^\alpha}{\alpha},$$

we reduce Eq. (15) into a nonlinear ODE of the form

$$cu' + kuu' + k^2\mu u'' + k^3\nu u''' + k^4\eta u^{(4)} = 0. \quad (17)$$

We suppose that Eq. (17) has a solution in the form given below

$$u(\xi) = \sum_{i=0}^{n} a_i\,\varphi^i(\xi), \quad (18)$$

where $a_i\,(i = 0,1,2,,\dots,n)$ are constants. Balancing the highest order derivative term $u^{(4)}$ and with nonlinear term $uu'$ in Eq. (17) we can obtain $n = 3$. So, we have

$$u(\xi) = a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3. \quad (19)$$

Substituting Eq. (19) into (17), using Eq. (12) and collecting all the terms with the same power of $\varphi^i$ $(i = 0,1,2,,\dots,7)$ and set them to be zero, a set of algebraic equations are obtained as follows

$$\begin{aligned}
\varphi^0:\ & c\sigma a_1 + k\sigma a_0 a_1 + 2k^3\nu\sigma^2 a_1 + 6k^3\nu\sigma^3 a_3 \\
& +2k^2\sigma^2\mu a_2 + 16k^4\eta\sigma^3 a_2 = 0,\\
\varphi^1:\ & 16k^3\nu\sigma^2 a_2 + 2k^2\sigma a_0 a_2 + k\sigma a_1^2 + 2c\sigma a_2 \\
& +2k^2\sigma\mu a_1 + 6k^2\mu\sigma^2 a_3 \\
& +120k^4\eta\sigma^3 a_3 + 16k^4\eta\sigma^2 a_1 = 0,\\
\varphi^2:\ & 60k^3\nu\sigma^2 a_3 + 8k^3\nu\sigma a_1 + k a_0 a_1 + 3k\sigma a_0 a_3 \\
& + 3k\sigma a_1 a_2 + 136k^4\eta\sigma^2 a_2 + 3c\sigma a_3 \\
& + c a_1 + 8k^2\eta\sigma a_2 = 0,\\
\varphi^3:\ & 40k^3\nu\sigma a_2 + 2k^2\mu a_1 + 18k^2\mu\sigma a_3 + 2k a_0 a_2 \\
& + k a_1^2 + 4k\sigma a_1 a_3 + 2k\sigma a_2^2 + 2c a_2 \\
& +40k^4\eta\sigma a_1 + 576k^4\eta\sigma^2 a_3 = 0,\\
\varphi^4:\ & 6k^3\nu a_1 + 114k^3\nu\sigma a_3 + 3a_3 + 3k a_0 a_3 + 3k a_1 a_2 \\
& +5k\sigma a_2 a_3 + 6k^2\mu a_2 + 240k^4\eta\sigma a_2 = 0,\\
\varphi^5:\ & 24k^3\nu a_2 + 3k a_1 a_3 + 2k a_2^2 + 3k\sigma a_3^2 + 12k^2\mu a_3 \\
& +24k^4\eta a_1 + 816k^4\eta\sigma a_3 = 0,\\
\varphi^6:\ & 60k^3\nu a_3 + 5k a_2 a_3 + 120k^4\eta a_2 = 0,\\
\varphi^7:\ & 3k a_3^2 + 360k^4\eta a_3 = 0
\end{aligned} \quad (20)$$

Solving the set of algebraic Eqs. (20) by using Maple 13, we have the following results

***Case 1.*** Consider

$$\begin{aligned}
& a_0 = \frac{-1}{64}\frac{11k\nu^3 + 64c\eta^2}{k\eta^2}, && a_1 = \frac{-15}{8}\frac{k\nu^2}{\eta},\\
& a_2 = -15k^2\nu, && a_3 = -120k^3\eta,\\
& \sigma = \frac{1}{64}\frac{\nu^2}{k^2\eta^2}, && \mu = \frac{1}{16}\frac{\nu^2}{\eta}.
\end{aligned} \quad (21)$$

***Case 2.*** Consider

$$\begin{aligned}
& a_0 = \frac{-1}{192}\frac{-5k\nu^3 + 192c\eta^2}{k\eta^2}, && a_1 = \frac{-5}{8}\frac{k\nu^2}{\eta},\\
& a_2 = -15k^2\nu, && a_3 = -120k^3\eta,\\
& \sigma = \frac{-1}{576}\frac{\nu^2}{k^2\eta^2}, && \mu = \frac{47}{144}\frac{\nu^2}{\eta}.
\end{aligned} \quad (22)$$

***Case 3.*** Consider

$$\begin{aligned}
& a_0 = \frac{-1}{1024}\frac{-15k\nu^3 + 1024c\eta^2}{k\eta^2}, && a_1 = \frac{-75}{128}\frac{k\nu^2}{\eta},\\
& a_2 = -15k^2\nu, && a_3 = -120k^3\eta,\\
& \sigma = \frac{-1}{1024}\frac{\nu^2}{k^2\eta^2}, && \mu = \frac{73}{256}\frac{\nu^2}{\eta}.
\end{aligned} \quad (23)$$

Using Case 1, Eq. (19) and the solutions of (13), we can find the following exact solutions of the conformable fractional generalized Kuramoto-Sivashinsky equation

$$u_1(x,t) = \frac{-1}{64}\frac{11k\nu^3 + 64c\eta^2}{k\eta^2} - \frac{15}{64}\frac{k\nu^3}{\eta^2}\,[\tan(\psi) + \tan^2(\psi) + \tan^3(\psi)], \quad (24)$$

$$u_2(x,t) = \frac{-1}{64}\frac{11k\nu^3 + 64c\eta^2}{k\eta^2} + \frac{15}{64}\frac{k\nu^3}{\eta^2}\,[\cot(\psi) - \cot^2(\psi) + \cot^3(\psi)], \quad (25)$$

$$u_3(x,t) = \frac{-c}{k} + 120\eta\left[\frac{k}{k\frac{x^\alpha}{\alpha} + c\frac{t^\alpha}{\alpha} + \omega}\right]^3, \nu = 0, \quad (26)$$

where $\psi = \frac{1}{8}\frac{\nu}{k\eta}\left(k\frac{x^\alpha}{\alpha} + c\frac{t^\alpha}{\alpha}\right)$ and $\omega$=const.

Using Case 2, Eq. (19) and the solutions of (13), we can find the following exact solutions of the conformable fractional generalized Kuramoto-Sivashinsky equation

$$u_4(x,t) = \frac{-1}{192}\frac{-5k\nu^3 + 192c\eta^2}{k\eta^2} + \frac{5}{192}\frac{k\nu^3}{\eta^2}\left[\tanh(\psi) - \tanh^2(\psi) + \frac{\tanh^3(\psi)}{3}\right], \quad (27)$$

$$u_5(x,t) = \frac{-1}{192}\frac{-5k\nu^3 + 192c\eta^2}{k\eta^2} + \frac{5}{192}\frac{k\nu^3}{\eta^2}\left[\coth(\psi) - \coth^2(\psi) + \frac{\coth^3(\psi)}{3}\right], \quad (28)$$

where $\psi = \frac{1}{24}\frac{\nu}{k\eta}\left(k\frac{x^\alpha}{\alpha} + c\frac{t^\alpha}{\alpha}\right)$.

From Case 3, we obtain other exact solutions of Eq. (15). Here, we omit them for the sake of simplicity.

***Remark 1.*** All solutions presented in this paper have been checked with Maple 13 by putting them back into the Eq. (15).

## 4. Conclusions

In this article, the sub-equation method has been successfully applied to find the exact travelling wave solutions of conformable fractional generalized Kuramoto-Sivashinsky equation. Based on the certain transformation, such CFPDE can be turned into ODE, the solutions of which can be expressed by a polynomial in $\varphi$, where $\varphi$ satisfies the Riccati equation $\varphi' = \sigma + \varphi^2$. With the aid of mathematical software, a variety of exact solutions consisting of hyperbolic functions and trigonometric functions for this CFPDE are obtained. Furthermore, the work shows that the algorithm is effective and can be used for many other CFPDEs in mathematical physics.

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