



Research Article

Effects of Circular Hierarchy Elements on Effective Elastic Properties of First Order Hierarchical Honeycombs

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Keywords

Circular hierarchy,
In-plane effective
properties,
Castigliano's second
theorem,
Finite element method,
Periodic boundary
conditions.

Abstract

The mechanical behavior of honeycombs with circular hierarchy under in-plane loading was investigated using analytical and numerical methods. To obtain the first-degree hierarchy, the combinations of the cell walls of the hexagonal base structure were replaced with smaller circular elements. Besides, a new parameter was obtained to control the material ability to resist deformations by applying different thickness status (heterogeneity). Effective Young's module and effective Poisson's ratio of the hierarchical structure were calculated by applying Castigliano's theorem. The effective shear modulus was also determined with the help of the equation for isotropic materials based on classical strength of materials. To easily interpret the effects of hierarchy and different thickness status on the non-hierarchical honeycomb, these mechanical properties were normalized with the mechanical properties displayed by the ordinary honeycomb. The effect of the scaling factor which controls the size of the circular geometry, and the heterogeneity factor which controls the wall thickness ratio between ones of base and hierarchical parts on the elastic properties are studied. The findings showed that by changing the scaling and heterogeneity factor, structures with 2 times the stiffness and 0.3 times the effective Poisson's ratio of an ordinary honeycomb having the same average density is obtained.

1. Introduction

Nowadays, the phenomenon of innovation has gained a considerable growing interest in every field, especially engineering. In order to meet the increasing demands in terms of performance and cost, the development of material technology has become especially crucial. One of the most important advancements in the field of material science and engineering stems from the exploration and manufacture of materials with cellular structure [1]. In addition to their superiority in terms of mechanical, thermal, electrical and acoustical properties, the most important advantage of cellular materials is their low density compared to other solid materials. Materials with high strength and high energy absorption properties even though they have low density and

therefore low weight due to the pores in them, are called cellular solid materials. While cellular materials can be artificially produced from polymers and metals, there are many examples of them in nature such as wood, leaf, animal bone, plant stem and honeycombs [2-8].

There is an abundant use of hierarchical cellular structures made of different materials, and of various sizes in nature. This can be exemplified by a human bone rich in hierarchy [9-14]. Inspired by nature, engineers, designers, and architects incorporate hierarchical structures in their work. So, the usage of the concept of hierarchy in a structure has come into being simultaneously in many different application areas such as Eiffel's Tower, Garabit Viaduct and Harbour Bridge [15,16]. The main purpose of introducing

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hierarchy is to improve the mechanical response of the structure without using different types of materials, changing the elastic properties of the material, and using more material to build up the structure. Hierarchical structures can be created by adding material to the most stressed parts, which are joints of the structure under loads. This method improves the efficiency of load carrying capacity [17].

Previous studies have explored the effect of hierarchy on mechanical properties of regular cellular structures [18,19, 20-22]. For example, Taylor and Smith [23] investigated the effect of square and triangular super and sub-structures on in-plane mechanical properties of a honeycomb cell model. In addition, they explored the negative Poisson's ratio effect observed as a result of hierarchy. Burgueno and Quagliata [24] studied hierarchical designs of beams made of bio-composite polymers. They found that hierarchical structures can compete with other traditional structures for load applications despite the decrease in the density of the structure. Fan et al. [25] have showed that two-dimensional hierarchical structures can be used to produce sandwiched composite panels with better stiffness, buckling strength, plastic collapse strength, and damage tolerance compared to solid materials. Zhang et al. have tried to make the cell walls multi-layered to increase the energy absorption capacity of the honeybee comb model, thus reporting that they have obtained a model with higher absorptivity [26]. Qing has worked on creating a three-dimensional hierarchical model of wood [27]. This model is used to represent the wooden structures at different scale levels. Transverse compression and shear collapse states of hierarchical corrugated truss structures were investigated by Kooistra [14]. The mechanical properties such as elastic buckling and yielding of the larger and smaller struts were examined. Ajdari et al. [28] studied the mechanical response of 2-dimensional honeycomb structures having hexagonal hierarchies using an analytical model validated with numerical simulations and experiments. They demonstrated that mechanical properties of the honeycomb structure such as the elastic modulus and Poisson's ratio can be modified by adding a hierarchy to the structure. With the information obtained from the researches made so far, in many engineering applications such as impact/blast energy absorption, thermal or sound isolation, the hierarchical honeycomb structures are preferred due to their low density and high effective strength.

In this study, to achieve lightweight and better-performing structures, we replaced the corners of the ordinary hexagonal lattice with smaller sized circles with a radius proportional to the edge length of the hexagonal base part by the factor of γ_1 (i.e., scaling factor or length scales). In this way, the hexagonal base structure has been introduced with the first-order hierarchy by circular hierarchy elements. We have also shown that the heterogeneity factor (i.e., wall thickness ratio, $\alpha_t = t_h/t_n$) influences the mechanical properties and can be used to tune them in addition to the hierarchy scaling factor. As a result of keeping overall density constant, the wall thickness for the homogeneous hierarchical structure is decreased compared to ordinary honeycomb's one. On the other hand, the wall thicknesses of the hierarchical and base parts of the heterogeneous structure change depending on the value of the heterogeneity factor besides the scale factor. Figure 1 shows the structural

hierarchies of honeycomb structures investigated in our work.

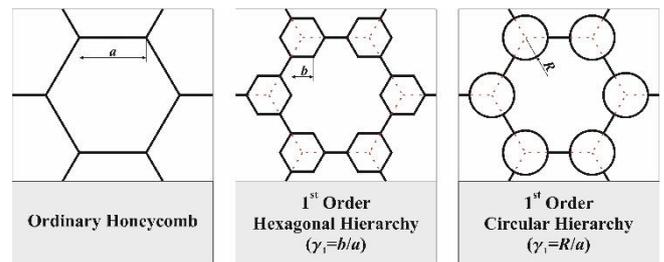


Figure 1. Ordinary honeycomb, first order hexagonal hierarchy and first order circular hierarchical honeycomb

Because we seek to determine the advantages of the hierarchical model investigated in this work, the boundary conditions, loading case, structural and material properties are adapted from Ajdari et al. [28] as a reference model. He set the wall length as 20 mm, relative density as 0.02, 0.06 and 0.10, and the material as ABS (Acrylonitrile Butadiene Styrene) polymer with Young's modulus of $E_s = 2300$ MPa and Poisson's ratio of $\nu_s = 0.3$ in his article. According to Gibson et al. [18], in-plane elastic properties such as stiffness and strength mainly depend on the bending deformation of cell edges under transverse loading. This assumption is particularly correct for slender beams that have a long length compared to their depth. To demonstrate this approach's accuracy, strain energy distribution was examined in the ordinary honeycomb using the parameters specified in Ajdari's study [28], and the results are shown in Table 1.

Table 1. Strain energy distribution of ordinary honeycomb

Parameters		Axial	Shear	Bending
E_s	2300 MPa	1.63E-3 kJ	1.7E-3 kJ	0.217 kJ
ν_s	0.3			
F	1 N			
a	20 mm			
t_n	1 mm			
		0.74 %	0.77 %	98.49 %

As seen in the table, the rate of the other reaction expressions in the strain energy stored in the structure cannot exceed 1.5 %, and the reaction moment that causes bending constitutes all the remaining energy. For this reason only the bending reaction moment is taken into account in all subsequent operations during the application of Castigliano's theorem.

The effective mechanical properties are a commonly used phenomenon for a heterogeneous body to be defined by benefiting from the relationship between the average stresses and the average strains. To compare these effective mechanical properties with the reference model, they are normalized with the ones which are the equations of $E_{ref} = 1.5E_s\rho^3$ and $\nu = 1$ governed Young's modulus and Poisson's ratio of ordinary honeycomb, respectively.

The measure of the hierarchy is specified according to the scaling factor which is the ratio of the specific dimension of the introduced hierarchy (edge length for hexagon geometry and radius for circular geometry), to the edge length of the ordinary hexagonal lattice as described in Figure 1 (i.e.,

$\gamma_1^{hex} = b/a$ and $\gamma_1^{circ} = R/a$. For a first-order circular hierarchical honeycomb, $0 \leq R \leq a/2$ so that, $0 \leq \gamma_1 \leq 0$. The case, $\gamma_1 = 0$, where there is no hierarchy corresponds to a regular honeycomb. The relative density of a structure with unit depth is the ratio of the area cell walls projected onto the plane of the structured to the full area covered by the unit cell, which is given by:

$$\rho_{circ}^{heter} = \frac{2t_n}{a\sqrt{3}} \left[1 + \gamma_1 \left(\frac{4\pi}{3} \alpha_t - 2 \right) \right] \quad (1)$$

For an ordinary honeycomb structure (i.e., when $\gamma_1 = 0$) $\rho = 2t/a\sqrt{3}$ and for a first-order circular hierarchical honeycomb structure with homogenous thickness $\rho_{circ}^{hom} = 2t[1 + \gamma_1(4\pi/3 - 2)]/a\sqrt{3}$ (i.e., the case of $\alpha_t = 1$). It is apparent in the relative density equation that the ratio, t/a has to decrease as the γ_1 increases to keep the relative density constant.

We have studied the in-plane effective elastic properties of the circular hierarchical honeycomb under plain stress condition analytically using Castigliano's theorem, and numerically using Finite Element Analysis (FEA). The analytical model for estimating the normalized effective Young's modulus, the normalized effective Poisson's ratio and the normalized effective shear modulus of the circular hierarchical honeycomb are presented in Section 2 and 3, respectively. The boundary conditions and loading case used to derive analytical model in Section 2 with the periodic boundary conditions applied in FEA are also summarized. The details of finite element analysis performed parametrically via Abaqus CAE for numerical investigation is given in Section 3. In Section 4, the in-plane elastic

properties of the circular hierarchical honeycomb are presented and compared with the hexagonal hierarchy studied by Ajdari et al. [28] and ordinary honeycomb. Also, the possibilities for further improvement of the mechanical performance of the structure is discussed.

2. Normalized Effective Young's Modulus

Alberto Castigliano, an Italian engineer, outlined a method to determine the displacement at a point in a body, referred to as Castigliano's second theorem. The theorem applies only to materials with linear-elastic behavior. According to the theorem, the displacement at a point to be determined is equal to the first partial derivative of the strain energy in the body to the external force acting at that point and in the force's direction. So, we use Castigliano's Second theorem to find out the deformations under the in-plane uniaxial loading case for estimating the effective properties of the circular hierarchical structures analytically. It has been shown that lattices having three-fold symmetry exhibit macroscopically isotropic behavior [29]. Therefore, we also assumed that the structure is made of an isotropic linear elastic material with elastic modulus, E_s and Poisson's ratio, ν_s in this work. Based on the assumption, it is possible to describe their mechanical behavior conveniently by using only two constants. Figure 2(a) shows the uniaxial loading and the boundary conditions of a finite size structure. Under these conditions, the contraction in the y -axis and the elongation in the x -axis of the structure are δ_y and δ_x , respectively.

Circular hierarchical honeycomb consists of the unit cell specified in Figure 2(b).

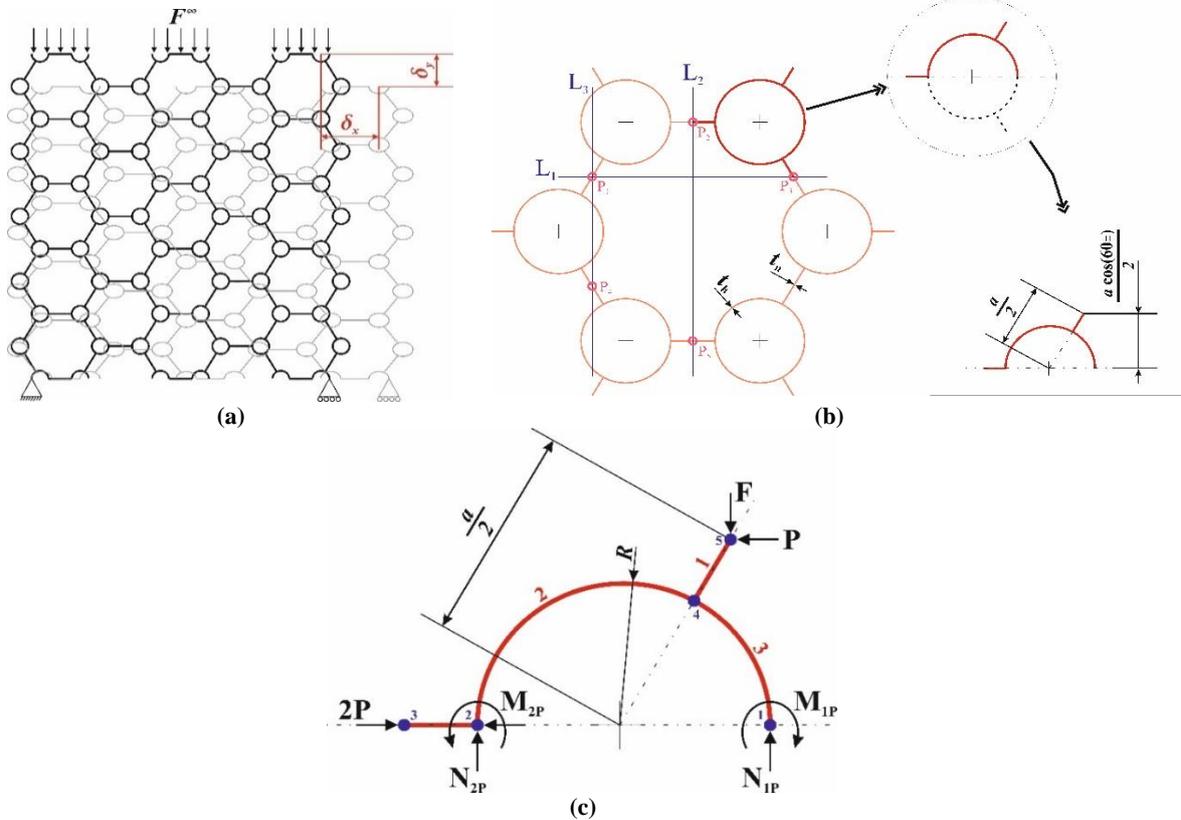


Figure 2. (a) Uniaxial-loading of the structure with a far-field force F^∞ (b) Symmetry axis and points of interest in a representative honeycomb (c) Free-body diagram of the structure used for analytical calculations

According to Saint-Venant's principle, the stress tends to reach a uniform value when we take the cross-section sufficiently far from the point where any external load is applied since the localized deformation effect caused by external loads vanishes. Consequently, assuming that the part of the structure we examined is far enough from the application point of the load, the far-field stress, $\sigma_{yy} = -(2/3)F/a$, occurs in the unit cell due to a load of F imposed in the y -direction. So, the average strain is $\varepsilon_{yy} = -4\delta_y/a\sqrt{3}$. The mechanical behavior of the entire structure can be studied by using subassembly, which is formed with L_1 and L_2 lines passing through the midpoints of the edges, as seen in Figure 2(b) [28].

The theoretical analysis presented in this section follows the approach presented in Ajdari et al. [28]. To make the analysis clear, it is important to examine the hierarchical honeycomb given in Figure 2(b). The midpoints of several edges of the given structure are named P_1 to P_5 . In the case of macroscopic normal stress, σ_{xx} and shear stress, σ_{xy} is the average force per unit length transmitted along a vertical line specified by L_2 . The net horizontal and vertical forces on this line are zero because only σ_{yy} is not zero. Moreover, they do not transmit the bending moments because it would remove the horizontal symmetry of the structure. Hence, under a non-zero macroscopic stress σ_{yy} , the edges cut by L_2 , are unloaded.

Considering the edges of the structure cut by L_1 , each bar is subjected to the force $F = -(3\sigma_{yy}a)/2$. Due to the symmetry in the structure, bending cannot occur in the struts cut by line L_1 , otherwise symmetry would be violated. This means that no bending moment is transmitted by the bars. Net horizontal force is zero across L_1 because $\sigma_{xy} = 0$. In the lower part of the structure cut by the L_1 line, a rightward force at P_1 is compensated by another leftward force at P_3 . Therefore, it can be stated that the forces at points P_1 , P_3 , and P_4 are vertical and F in magnitude.

Figure 2(c) presents the free body diagram of the subassembly of the circular hierarchical structure exposed under loading case as shown. The lower part of the subassembly causes the force and moment reactions (N_{1P} , M_{1P} , N_{2P} , and M_{2P}) at points 1 and 2. The vertical forces at points 1, 2, and 4 are used to determine the effective elastic modulus, whereas horizontal dummy forces (no actual load) are placed at 2, 3 and 4 in order to determine the lateral displacement using Castigliano's Method, to obtain the Poisson's ratio. By using vertical force equilibrium and moment balance equations for the subassembly, N_{2P} and M_{2P} can be written as a function of N_{1P} , M_{1P} , and F . To calculate the bending energy stored in the structure, the strain energy of the curved and linear beam elements that make up the subassembly should be calculated separately and summed as shown in the following expression: $U(F, N_{1P}, M_{1P}) = \int M^2/(2E_s I) dx + \sum \int (M^2 R)/(2E_s I) d\theta$, where M is the bending moment at location x for the linear beam, and θ for the curved beam; R is the radius of the curved beam, E_s is the elastic modulus of the structure material, I is the cross-section area moment of inertia (The moment of inertia which is calculated using the wall thickness t for the unit depth structure with the rectangular

section; i.e., $I = t^3/12$). U is a quadratic function of the quantities F , N_{1P} , and M_{1P} .

Because of the symmetry of the subassembly, it is assumed that there is no vertical displacement and rotation at point 1. So that, the equations $\partial U/\partial N_{1P} = 0$ and $\partial U/\partial M_{1P} = 0$ are obtained by application of Castigliano's method. These two equivalence relations allow us to derive the functions of N_{1P} and M_{1P} expressions in terms of F : $N_{1P} = F(0.529 + 0.138/\gamma_1)$, $M_{1P} = aF(-0.029 + 0.253\gamma_1)$. Using the equation of $\delta_y = dU/dF$ at point 4 allows finding the vertical displacement of the subassembly. We can express the displacement as a function of the F variable by substituting above N_{1P} and M_{1P} relations, and finally $\delta_y = \sqrt{3}Fa^3/(72E_s I_n f(\gamma_1, \alpha_t))$ is obtained, where $(\gamma_1, \alpha_t) = \sqrt{3}/[(0.75 - 4.5\gamma_1 + 9\gamma_1^2 - 6\gamma_1^3) + \alpha_t^{-3}(0.993\gamma_1 - 6.095\gamma_1^2 + 10.992\gamma_1^3)]$. The ratio of the average stress ($-2F/3a$) and the average strain ($-4\delta_y/a\sqrt{3}$) allows us to compute the effective elastic modulus of the model:

$$E/E_s = (t_n/a)^3 f(\gamma_1, \alpha_t) \quad (2)$$

For homogeneous thickness ($t_n = t_h$), α_t is equal to 1. If the heterogeneity factor is greater than 1, then the hierarchy part of the structure becomes thicker than the base part ($t_h > t_n$). Otherwise, the more strength region is the base part of the body ($t_n > t_h$).

To obtain the maximum value of the normalized effective elastic modulus, the term t_n/a in Eq. (2) should be converted to its equivalent in terms of α_t and γ_1 by using Eq. (1) and heterogeneity factor. Then, Taylor's theorem for a function of two variables is used, as given in Eq. (3), to find the local maximum of the resulting expression of the normalized effective elastic modulus.

$$\frac{\partial^2(E/E_s)}{\partial \gamma_1^2} \frac{\partial^2(E/E_s)}{\partial \alpha_t^2} - \frac{\partial}{\partial \alpha_t} \left(\frac{\partial(E/E_s)}{\partial \gamma_1} \right) > 0 \quad (3)$$

The equation gives the stationary point $\gamma_1 = 0.279$ and $\alpha_t = 0.701$, and then by substituting these values, $E/E_s = 3.036\rho^3$ is obtained. The stiffness of the first-order circular hierarchical honeycomb is slightly greater than twice of the stiffness of the ordinary honeycomb model [18], and is also marginally stiffer than the first-order hexagonal hierarchy [28].

To validate the analytical results, we simulated the mechanical response of the structure using finite element analysis (FEA). Two-dimensional hierarchical honeycombs were modeled using Abaqus 6.14 (SIMULIA, Providence, RI), where the structure is modeled with the BEAM22 quadratic beam elements which take axial and shear deformations into account in addition to the bending deformation. However, the contribution of axial and shear deformations to the response of the structure is negligible, when the beams are slender i.e. the ratio, a/t is large. The relative density of the structure is fixed by adjusting the thickness of the rectangular cross-section with the unit depth of the beams. It was also assumed that the structure is made of ABS polymer (Acrylonitrile Butadiene Styrene) with Young's modulus $E_s = 2300$ MPa, $\nu_s = 0.3$.

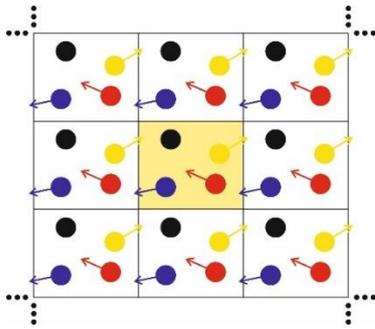


Figure 3. Two-dimensional schematic representation of the idea of PBCs

Periodic boundary conditions (PBCs) comprise a set of boundary conditions that allow us to split any large (infinite) system into small periodic parts called a unit cell and deal with discretely. PBCs are a consistently preferred approach for mathematical models and numerical investigations [30,31]. The idea of PBCs suggests a perfect two-dimensional link between these repeating unit cells of which the periodic cellular structure is composed. This state is shown schematically in Figure 3. With the homogenization method based on strain energy studied by many researchers and prescribed periodic boundary conditions, the structure's effective mechanical properties can be determined from its unit cell. Finally, to reduce computational time and simplify the finite element simulation process, it is reasonable to use an RUC with orthogonal lattice instead of the whole body, as shown in the figure, utilizing PBCs and performing the numerical analysis parametrically with Python software. In this study, PBCs were applied during the analysis to reduce the model size and processing time. Furthermore, we used the Repetitive Unit Cells (RUC) in Figure 4 with orthogonal lattice vectors (a_1, a_2) in FE analysis due to allowing a more straightforward application of periodic boundary conditions.

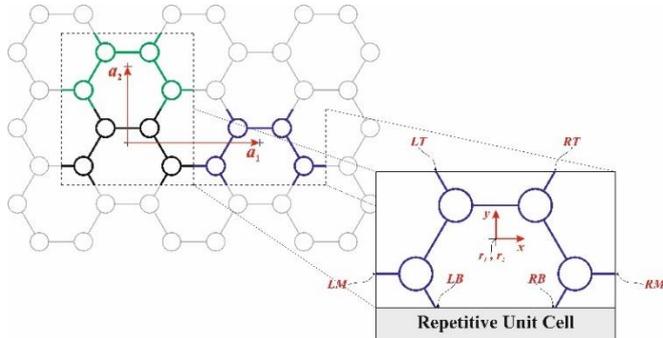


Figure 4. Repetitive Unit Cells and lattice vectors of the structure, and the points for periodic boundary conditions

A set of kinematic boundary conditions is applied to the periodic hierarchical structure as seen Figure 4 such that it results in the displacement field can be denoted as in Table 2. All nodes lying along the edges shown with the dashed lines in Figure 4(b) are connected to each other. In this way, the model behaves as if it is infinitely long and wide cellular structure but free to strain laterally [32]. As a result, we obtained an infinite cellular structure using this RUC, and eliminated the size effect (i.e., a size-independent structure).

Table 2. Constraints used to apply periodic boundary conditions

Constraints for PBCs	Boundary Conditions
$u_{LT} - u_{LB} = \bar{\epsilon}_i(x_{LT} - x_{LB}) = \bar{\epsilon}_i \Delta x = u_{r1}$	$u_{r1} = 0$
$u_{RT} - u_{RB} = \bar{\epsilon}_i(x_{RT} - x_{RB}) = u_{r1}$	$u_{r2} = free$
$u_{RM} - u_{LM} = \bar{\epsilon}_i(x_{RM} - x_{LM}) = u_{r2}$	$v_{r1} = cons.$
$v_{LT} - v_{LB} = \bar{\epsilon}_j(y_{LT} - y_{LB}) = v_{r1}$	$v_{r2} = 0$
$v_{RT} - v_{RB} = \bar{\epsilon}_j(y_{RT} - y_{RB}) = v_{r1}$	$u_{LB} = v_{LB} = 0$
$v_{RM} - v_{LM} = \bar{\epsilon}_j(y_{RM} - y_{LM}) = v_{r2}$	

In order to facilitate the comparison with the reference body, the effective elastic modulus values of the circular hierarchical honeycomb are normalized with that of ordinary honeycomb, following Ajdari et al. [28]. The normalized effective Young's modulus of the first-order circular hierarchical honeycombs for all values of γ_1 between 0 and 0.5 is given in Figure 5.

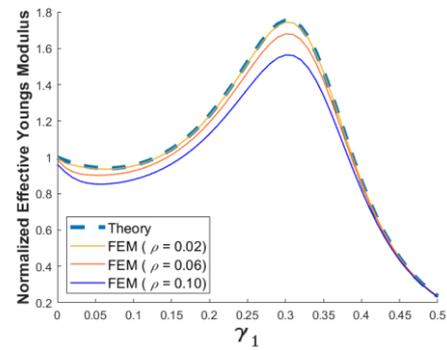


Figure 5. Normalized effective Young's modulus of the honeycomb structure with circular hierarchy for different relative densities with respect to γ_1

Neglecting the shear and axial deformations in the analytical model leads to discrepancies between theoretical and numerical results for higher relative-densities, i.e., for thicker cell walls where shear and axial deformations become essential. On the other hand, it is observed that the theoretical approach is entirely independent of the relative density. Though, a good agreement between numerical and analytical results is observed for low relative densities, see Figure 5. (a good approximation is only seen for the lowest density [32]).

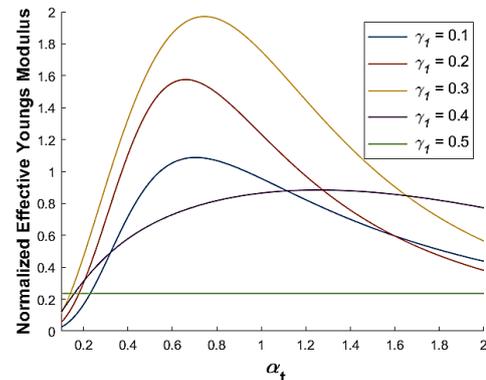


Figure 6. Normalized effective Young's modulus of the honeycomb structure with circular hierarchy with respect to α_t

Figure 6 represents how the 2-dimensional curves representing different γ_1 values are affected by the α_t change. In this interaction, the most sensitive γ_1 curve to α_t is $\gamma_1=0.3$ curve. In this case, it is seen that approximately 60 percent of the edge length of the base part of the building is covered with circular geometry, which is the hierarchy element, and the remaining part remains the base structure, making the structure more sensitive to the thickness ratio. The reason why the curve, $\gamma_1=0.5$, is never affected by the change of α_t value is that there is no base structure in this γ_1 value.

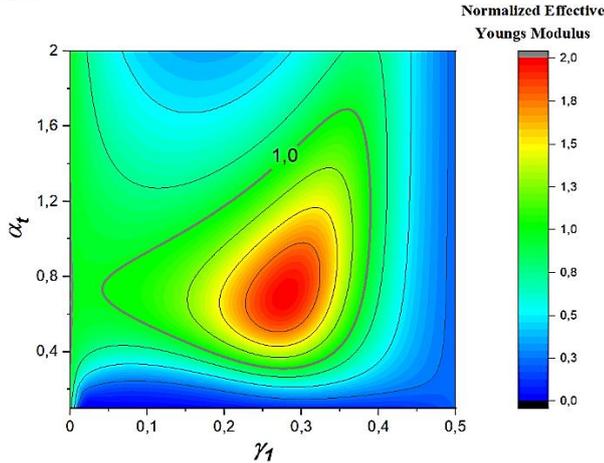


Figure 7. Contour plot of normalized effective Young's modulus as a function of the scaling factor and heterogeneity factor

Figure 5 shows that the circular hierarchical structure with homogeneous thickness and a scaling factor of $\gamma_1=0.31$ has stiffness nearly 1.8 times of the ordinary honeycomb. The contour map in Figure 7 demonstrates that when both the scaling factor and the heterogeneity factor are left as free parameters, a maximum stiffness of approximately 2.1 times that of base structure can be obtained when $\gamma_1=0.28$, and $\alpha_t=0.7$.

According to Figure 6, the following comments can be made about the values that α_t can take. First of all, it should be known that α_t can never be zero since it is the thickness ratio. The Contour graph shows that for the α_t values between 0.1-0.2, the structure's sensitivity to the scale factor disappears. This is because the hierarchy part is very thin. Thus, the increase in the ratio of this part in the structure will not affect the results. The values between 0.3 and 0.4 indicate the transition zone. It displays ordinary honeycomb properties for a specific γ_1 range at approximately 0.4 and 1.6. Between these two values, the stiffness of the structure increases non-linearly as α_t increases initially and decreases to the reference structure's stiffness. With different γ_1 values at values higher than 1.6, we always obtain a structure with a lower strength than ordinary structure.

If the same situation is interpreted for γ_1 , the structure's stiffness is no longer affected by the ratio of the thickness of the base and the hierarchy parts at the values of γ_1 in the range of 0.45-0.5. The reason is that the length of the base part of the body in this value range is very short compared to that of the other part. It can be called the transition zone for the γ_1 range of 0.4-0.35. The stiffness value obtained for the 0.35 and 0.05 values of γ_1 will be the same as ordinary honeycomb in the α_t values in a specific range. For a widen

α_t value range, the young's module of the structure is more than the reference structure between these two γ_1 values. The structure behavior in the range of 0-0.05 is almost the same as specified by 0.45-0.5. In this case, the base part's length will be larger than the one of the hierarchy part. For the specific area definition where the structure exhibits highly stiff behavior, the values of α_t and γ_1 should remain between 0.6-0.8 and 0.25-0.3, respectively. As mentioned earlier, the highest young's module value is reached when $\gamma_1=0.28$, and $\alpha_t=0.7$. The conclusion to be drawn in general is that ordinary honeycomb boundaries, reminiscent of a triangular geometry, are indicated by a gray closed-curve. The structure will be stiffer than the reference body as long as it is in this triangular region. Approximately, in the center of the area of this region, the structure will turn into the stiffest state.

3. Normalized Effective Poisson's Ratio & Normalized Effective Shear Modulus

For a complete identification of effective mechanical properties of the structure modeled as linear elastic, an equation in which the Poisson's ratio can be represented in terms of γ_1 and also α_t is needed. Therefore, Castigliano's second theorem is used to compute the lateral deformation of the structure subjected to uniaxial loading. The horizontal forces in Figure 2(c) are used as the dummy force in the energy expression used in Castigliano's theorem. Following a procedure similar to the derivation of the elastic modulus equation, axial and shear deformation of the beams are also ignored in this analysis. N_{2P} and M_{2P} can be written as functions of N_{1P} , M_{1P} , P , and F by using equations of equilibrium of subassembly. The summation of bending energy in all beams express the total energy stored in the subassembly under the loading case. So, $U(F, P, N_{1P}, M_{1P}) = \int M^2 / (2E_s I) dx + \sum \int (M^2 R) / (2E_s I) d\theta$. According to the assumption of no displacement and no rotation at nodes 1 and 2, $\partial U / \partial N_{1P} = 0$, and $\partial U / \partial M_{1P} = 0$ can be written. These two conditions enable us to write N_{1P} and M_{1P} in terms of P and F . This allows us to express the bending energy of the subassembly as a function of P and F , $U = U(P, F)$. Setting the dummy force P to zero, the lateral displacement of the subassembly is computed from $\delta_x^F = (\partial U / \partial P)_{P=0}$ and the vertical displacement from $\delta_y^F = (\partial U / \partial F)_{P=0}$. The initial dimensions of the subassembly are $3a/4$ and $a\sqrt{3}/4$ in the x - and y -directions, respectively. Therefore, the effective Poisson's ratio is obtained as $\nu = -\epsilon_x / \epsilon_y = -\delta_x^F / \sqrt{3}\delta_y^F$, which gives:

$$A = \gamma_1(-0.1655 + 1.0159\gamma_1 - 1.6407\gamma_1^2) + \alpha_t^3(-0.125 + 0.75\gamma_1 - 1.5\gamma_1^2 + \gamma_1^3) \quad (4)$$

$$B = \alpha_t^3(-0.5 + \gamma_1)^3 + \gamma_1(-0.1655 + 1.0159\gamma_1 - 1.8319\gamma_1^2) \quad (5)$$

$$\nu = A/B \quad (6)$$

For homogeneous thickness ($t_n = t_h$), α_t is equal to 1. To obtain the maximum value of the effective Poisson's ratio, Taylor's theorem for a function of two variables is used as given in Eq. (3). The value of the effective Poisson's ratio is $\nu = 1$ at $\gamma_1 = 0$, and the minimum value 0.2996 is reached

at $\gamma_1 = 0.326$ and $\alpha_t = 6.57 \times 10^{-6}$. However, this thickness ratio is too small for practical applications. Therefore, we consider physically reasonable value, $\alpha_t = 0.1$ which provides a Poisson's ratio, $\nu = 0.3$ that is very close to the minimum. Theoretical results were verified with finite element analysis of the periodic structure subjected to uniaxial loading. The normalized effective Poisson's ratio of the first-order circular hierarchical honeycombs with homogeneous thickness for values of γ_1 between 0 and 0.5 shown in Figure 8.

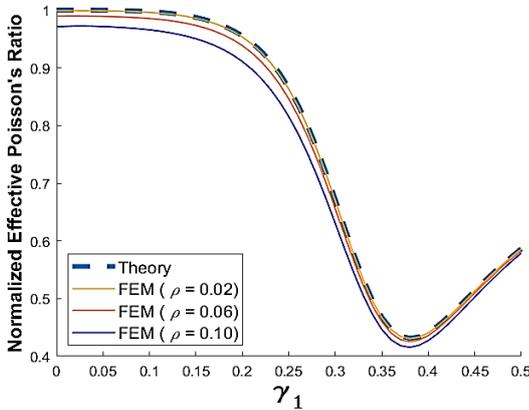


Figure 8. Normalized effective Poisson's ratio of the honeycomb structure with circular hierarchy for different relative densities with respect to γ_1

The results obtained by numerical analysis converge the results obtained theoretically as the relative density value decreases. Among the results obtained from these two different approaches, the reason for the differences evident with increasing ρ can be evaluated as neglect of the effects of axial and shear deformations in the theoretical approach. The adverse effects of this approach, which simplifies the computational process, decreases with the increase of γ_1 values. So, the following conclusion can be made; the curved beam elements' sensitivity in the structure's hierarchy parts to the axial and shear deformation effects is less than the linear beams in the base part. As a result of the dominant hierarchy elements in the structure with increasing γ_1 values, neglecting these mentioned deformation types does not affect our results. Additionally, Figures 8 shows that the lowest effective Poisson's ratio, 0.43 for the circular hierarchical honeycombs in the homogeneous thickness case ($\alpha_t = 1$) are achieved when γ_1 is 0.3 and 0.38, respectively.

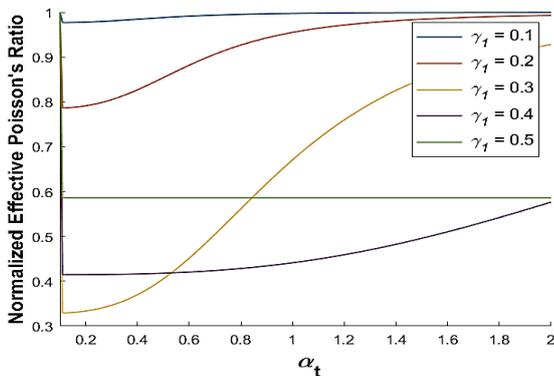


Figure 9. Normalized effective Poisson's ratio of the honeycomb structure with circular hierarchy with respect to α_t

It can be directly seen from the curves in Figure 9 that for $\gamma_1 = 0.1$ and $\gamma_1 = 0.5$, the normalized effective Poisson's ratio of the structure does not change depending on the value of α_t . In case the γ_1 value is 0.1, that is, the hierarchy part is almost absent in the structure, and the γ_1 value is 0.5, that is, the base part is not located in the structure, the body becomes unaffected by the values of the α_t .

Another critical parameter when thinking about the mechanical properties of an isotropic material is the shear module. The shear modulus is a measure of the resistance of a structure against distortive effects when it experiences the forces parallel to the applied surface while they are opposite to each other. In fact, two parameters are sufficient to determine the mechanical properties of isotropic materials. Therefore, the third parameter can be expressed as a function of the other two ones. However, the α_t and γ_1 -related changes in the parameters of the hierarchical structure are not similar. Thus, it is difficult to predict the replacement of the shear module by considering Young's modulus and Poisson's ratio. Hence, the rigidity of the structure under shear loads is obtained using the shear modulus equation for isotropic materials based on classical strength of materials. Shear modulus is governed by:

$$G = \frac{E}{2(1 + \nu)} \quad (7)$$

In Figure 10(a), the normalized effective Poisson's ratio values of the circular hierarchical structure are given. As seen in the figure, the effective Poisson's ratio value of the hierarchical structure decreases with the increase of γ_1 . With the increase of the radius of the hierarchy element, that is, with the increase of γ_1 , the structure becomes more resistant to lateral contraction caused by axial deformation. The physical meaning of ordinary honeycomb's normalized effective Poisson's ratio value is 1 is interpreted as follows; when the length of the structure under loading is extended or compressed by 1-unit in the direction of the applied force, the same amount of change perpendicular to the force direction is observed. However, in the region indicated by the blue-colored area, the approximate normalized effective Poisson ratio of the structure is between 0.3-0.4. It means that the effect of 1-unit change in the direction of force causes about 0.3-0.4-unit change in the lateral direction. Thus, it can be said that the hierarchy increases the stiffness of the structure against lateral deformation caused by longitudinal deformation in the direction of the force. The contour graph in Figure 10(a) shows that a minimum normalized effective Poisson's ratio of approximately 0.3 times that of base structure can be obtained when $\gamma_1 = 0.33$ and $\alpha_t = 0.1$.

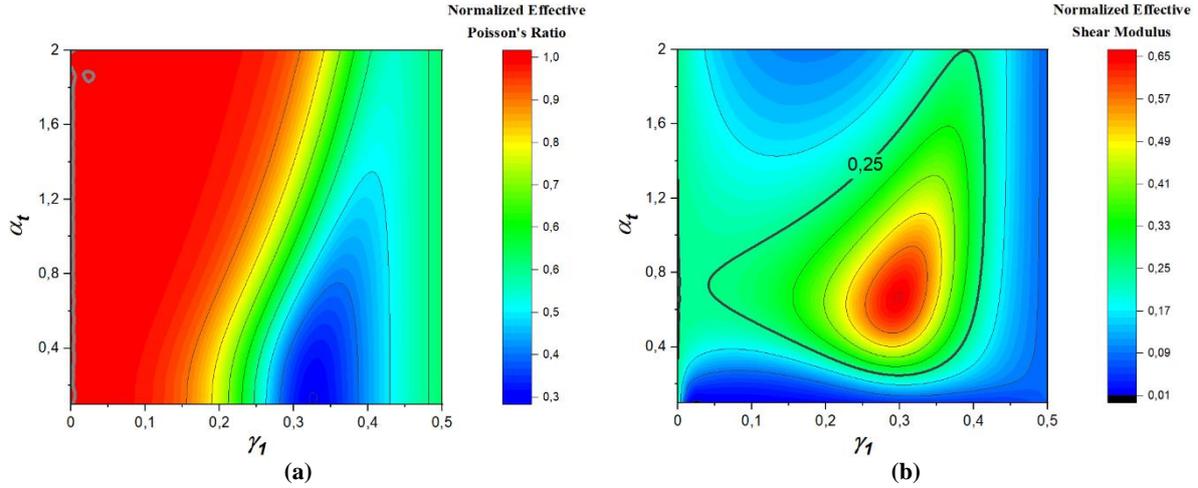


Figure 10. (a) Contour graph of normalized effective Poisson's ratio, and (b) normalized effective shear modulus as a function of the scaling factor and heterogeneity factor

On the other hand, the values of the normalized effective shear modulus of the circular hierarchical structure are given in Figure 10(b). Using Eq. (7), the value of the normalized effective shear modulus of the ordinary honeycomb structure is calculated as 0.25. A gray closed-curve indicates this value of the reference structure on the Contour graph, as in the contour graph given for the normalized effective Young's modulus, the region where the shear modulus is stiffer against the shear loadings than the reference structure has the appearance of approximately triangular geometry. The contour graph in Figure 10(b) represents that a maximum normalized effective shear modulus of 0.65 can be obtained when $\gamma_1 = 0.3$, and $\alpha_t = 0.7$.

Thus, it is clear that the mechanical performance is improved by using different thickness method when compared to the results of the heterogeneous case.

4. Conclusion

Figure 11 demonstrates all the effective elastic properties of the hierarchical structure. Normalized versions of these terms were given in the previous sections. Normalized terms allowed us to compare the hierarchical structure to ordinary honeycomb. Nevertheless, it is known that the actual values of these effective properties are also required during the design for designers. It is seen that the effective Young's modulus and the effective shear modulus in Figure 11(a)-(c) reach the maximum value at the same α_t and approximately the same γ_1 . As can be clearly seen from these graphs, a steady upward trend in Young's and shear modulus values is not observed when α_t increases.

Besides, the hierarchical and the base parts of the structure are quite thin compared to the other part when the alpha ranging from 0.1 to 2 get the extreme values. Therefore, the structure becomes the weakest in terms of the effective Young's and shear modulus as expected. And the minimum and maximum values of γ_1 are non-hierarchy ($\gamma_1 = 0$) and full-hierarchy ($\gamma_1 = 0.5$) structures, respectively.

In the non-hierarchy state, the value read from the curve represents the ordinary honeycomb structure's properties. In the full-hierarchy state, all variables lose their effect on the result, and all curves merge on a specific value. This is due to the complete disappearance of the base part of the structure. In Figure 11(b), a decreasing trend is recognized in the effective Poisson ratio due to the decrease in α_t . For the effective Poisson ratio, the structure shows the same behavior with the other mechanical properties at the maximum and minimum values of the γ_1 . Consequently, these three elastic mechanical properties reach maximum and minimum values, as desired, at approximately the same γ_1 value. It is between 0.3-0.35.

For the effective Young's modulus and the effective shear modulus curves, the position of the peak point in each α_t curve shifts to the right on the γ_1 axis as the α_t value increases. After $\alpha_t = 0.7$, the shifting accelerates. This situation is seen as a stable right shifting movement following the increase of α_t value in the effective Poisson's ratio curve. Except for a few α_t values ($\alpha_t = 0.6-0.9$), as the γ_1 increases, the strength values of the structure at all α_t values decrease first and then start to increase again. The amount of the rigidity reduction is mostly at $\alpha_t = 0.1$. When the structure transforms from non-hierarchy ($\gamma_1 = 0$) to full-hierarchy ($\gamma_1 = 0.5$), its stiffness decreases by 76.5%; this reduction rate is 70.4% for shear modulus and 41.4% for Poisson's ratio. Among the points that show the maximum and minimum values that γ_1 can take, the ideal situation differs for the three specified mechanical property terms. Therefore, it is necessary to work within a specific range in order to obtain the required mechanical properties. The range is between $\gamma_1 = 0.28$, which is the maximum of Young's module, and $\gamma_1 = 0.33$, which is the minimum of Poisson's ration. We call it as the ideal design range for the first-order circular hierarchical honeycomb.

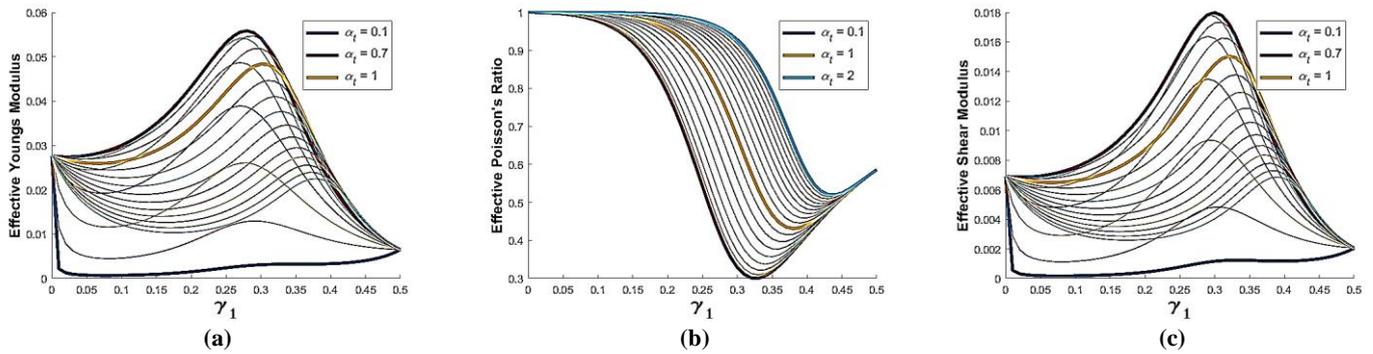


Figure 11. (a) Normalized effective Young's modulus, (b) Normalized effective Poisson's ratio, and (c) Normalized effective shear modulus of the circular hierarchical honeycomb as a function of the heterogeneity factor with respect to γ_1

Figures 12 and 13 demonstrate the normalized effective elastic modulus and the normalized effective Poisson's ratio for different heterogeneity levels. The results show that a relatively wide range of normalized elastic modulus and effective Poisson's ratio can be obtained by tailoring the structural dimensions of the honeycombs or changing the hierarchy geometry. Furthermore, it can be seen that the first-order hexagonal hierarchical honeycombs have an effective stiffness up to 2 times of the ordinary honeycomb [28]. It is possible to achieve an effective elastic modulus up to 2.1 times that of the ordinary honeycomb using circles instead of hexagons as hierarchy geometry and by setting different thicknesses for the base and hierarchy parts of the body.

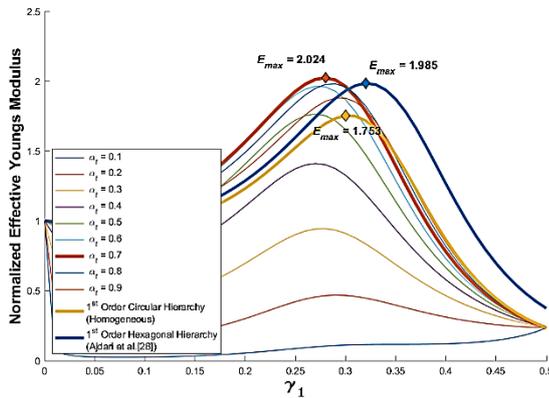


Figure 12. Normalized effective Young's modulus of the first-order circular hierarchical honeycomb for different heterogeneity levels, and the first-order hexagonal hierarchical honeycomb with respect to γ_1

Now that, maximum values of the bending moments in each cell wall, which influence the effective properties of the structure, occur at the corners of the honeycomb. The in-plane elastic properties tend to increase as the amount of material located at the corners of the structure is more than that found in the middle parts of the cell walls. That's why the hierarchy is a widespread phenomenon for cellular structure. This situation can be seen in Figures 12 and 13, where the findings we obtained in our study are shown. We see an increase in stiffness up to $\gamma_1 = 0.3$. The reason is an increase in the amount of material located at the corners of the body, as mentioned above. The increase can be calculated numerically by $(2\pi t_{final} - 3t_{initial})\alpha\gamma_1$.

On the other hand, the numerical value of the change in the amount of material located at the middle part of the structure is found with the equation $3a(t_{final} - t_{initial})/2 - \alpha\gamma_1 t_{final}$, where t_{final} is the wall thickness of the reference structure. As can be easily realized from the equations, with the increase of γ_1 value, the body's corner parts get sufficiently durable, and the middle part becomes weak. Considering the homogeneous thickness condition resulting from this situation, while the stiffness increases up to a specific γ_1 value, it starts to decrease after a pick point. In case of heterogeneity, it is seen that by keeping the γ_1 constant, α_t increases the stiffness to a particular value and then decreases it. The behavior of the structure, as described, is explained similarly to the state in the homogeneous system.

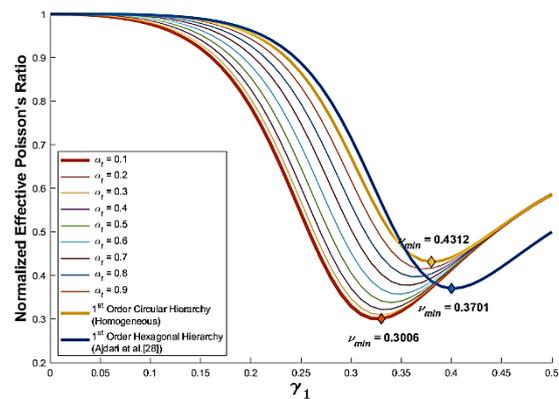


Figure 13. Normalized effective Poisson's ratio of the first-order circular hierarchical honeycomb for different heterogeneity levels, and the first-order hexagonal hierarchical honeycomb with respect to γ_1

Figures 12 and 13 demonstrate that; by choosing circular geometries as the hierarchy element, we can obtain a mechanically better structure from the reference body, which is the ordinary honeycomb. However, we get a worse design than the hierarchical structure with hexagon preferred by Ajdari as a hierarchy element. When we consider the hierarchy and base parts of the body separately in terms of wall thickness, which we call heterogeneity in our study, we could obtain a structure with a lower stiff and Poisson ratio from the hexagonal hierarchical structure concerning the mechanical properties. With heterogeneity, an improvement of 15.5% in Young's modulus and 30.3% in Poisson's has been achieved. Further maximization of features can be

possible by also varying hierarchy geometries. It would be useful if the hierarchy elements are in different thickness conditions to get maximum mechanical performance. This could be achievable by appropriate adjustment of the cell thickness in different parts of the structure.

Acknowledgements

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