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An Efficient Approach for Solving the Linear and Nonlinear Integro Differential Equation

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Keywords

Integro differential equation, Fredholm integro differential equation, Voltera integro differential equation, Variation of parameters method, Error.

Abstract

In this article, we are utilizing variation of parameters method in integro differential equations. These equations assume a dynamic job in demonstrating of numerous physical developments in science and engineering. Motivated and inspired by these facts, we are applying sucessfully analytic technique to solve first-order Fredholm Integro Differential Equations (FIDE) and Voltera Integro Differential Equation (VIDE). The recommended technique is applied with no discretization, linearization, Perturbation, transformation, preventive presumptions and is liberated from Adomian's polynomials. Various models are given to determine the legitimacy and appropriateness of the proposed technique.

1. Introduction

Lately, there has been a developing interest for the Integro-differential equations (IDE), which are a mix of differential and integral equations. The result of IDE have a significant job in the fields of science and designing. At the point when a physical system is demonstrated under the differential intellect, it at last gives a differential equation (DE), a integral equation (IE) or an Integro-differential equation (IDE). IDE is an important branch of modern mathematics and arises frequently in many applied areas which include engineering, mechanics, physics etc. In this paper, consider the Integro-differential equations of the sort of Eq. (1)

$$Z^n(x) = f(x) + \int_a^b K(x,s)Z(s)\,ds, \qquad (1)$$

with the initial conditions

$$Z^{(k)}(a) = b_k,\ k = 0,1,2,\dots,(n-1) \qquad (2)$$

where $Z^n(x)$ is the nth derivative of the unknown function $Z(x)$, $K(x,s)$ is the kernel of the equation, $f(x)$ is the are the analytic function and a,b,k are the appropriate constants. As of late, several authors focus on the improvement of numerical and logical strategies for Integro-differential equations. For example, K. Maleknejard et al. using Haar function technique [1], E. Yusufoglu (Agadjanov) using homotopy perturbation technique [2], S.Q Wang et al. using Variation iteration technique [3], M. Ghasemi et al. utilizing He's HPM technique [4], A.A. Hamoud et al. utilizing MADM [5], A.A. Hamoud et al utilizing numerical technique [6], Manafianheris using LADM [7], Nahid et al. utilizing combined homotopy analysis transform method [8], Jackreece et al. utilizing Taylor series and Variational iteration technique [9] and Ali et al. utilizing HPM [10].

In addition, variation of parameters technique evacuates the higher order derivative term from its iterative plan which is clear favourable position over the variational cycle strategy as the term may reason for rehashed calculation and computations of unneeded terms, which expends both the time and exertion, in a large portion of the cases. Consequently, variation of parameters strategy has diminished part of computational work required because of this term when contrasted with some other existing strategies utilizing this term which is away from of proposed procedure over them. Henceforth, variation of parameters strategy gives a more extensive and better relevance as contrast with other old style procedures. Noor et al. [11] demonstrated variation of parameters technique to resolve some higher order differential equations. Zaidi et al. [12] utilized variation of parameters technique for thin flow film. In this paper, we have applied variation of parameters strategy to understand linear and nonlinear Fredholm Integro-differential equation (FIDE) and of Voltera Integro-differential equations (VIDE). In Section 2, we talk about the derivation of the variation of parameters strategy to pass on the thought. A few models are considered in Section 3 to show the usage and proficiency of the variation of parameters strategy. Results are very reassuring and may invigorate further research in the utilizations of this technique.

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2. Variation of Parameters Method (VPM)

To derive the main concept of the VPM [13], we assume the general Eq. (3)

$$L(w) + N(w) + R(w) = f(x) \quad a \leq x \leq b, \tag{3}$$

where L, N is a linear and non-linear operators. L, R is a linear differential operator, but L has the highest order than R, f(x) is a source term in the given domain $[a, b]$. By utilizing the VPM, we have the pursuing solution of the Eq. (3)

$$w(x) = \sum_{i=0}^{k-1} \frac{C_{i+1}x^i}{i!} + \int_0^x \lambda(x,\tau)\big(-N(w)(\tau) - R(w)(\tau) + f(\tau)\big)d\tau\,, \tag{4}$$

where k represent the order of given differential equation (DE) and C_i where $i = 1,2,3, \ldots.$ are unknowns.

$$\sum_{i=0}^{k-1} \frac{C_{i+1}x^i}{i!} \tag{5}$$

For homogeneous solution which is taken by $L(w) = 0$,

The another part which is obtained by Eq. (4) by using VPM.

$$\int_0^x \lambda(x,\tau)\big(-N(w)(\tau) - R(w)(\tau) + f(\tau)\big)d\tau, \tag{6}$$

Here $\lambda(x,\tau)$ is a Lagrange multiplier, that expel the progressive use of integrals in the iterative scheme and it depending on the order of equation. Commonly the pursue definition is utilized to find the value of the multiplier $\lambda(x,\tau)$ from

$$\lambda(x,\tau) = \sum_{i=1}^{k} \frac{(-1)^{i-1}\tau^{i-1}x^{k-i}}{(i-1)!(k-1)!} = \frac{(x-\tau)^{k-1}}{(k-1)!}\ , \tag{7}$$

Here k is the order of the given DE and it varies for different values of k. we have the pursue conditions

k =1, $\lambda(x,\tau) = 1$,
k = 2, $\lambda(x,\tau) = (x-\tau)$,
k = 3, $\lambda(x,\tau) = \frac{x^2}{2!} + \frac{\tau^2}{2!} - \tau x$,
.
.

Therefore, we utilize the pursue iterative scheme for solving Eq. (8)

$$w_{n+1} = w_0 + \int_0^x \lambda(x,\tau)\big(-N(w)(\tau) - R(w)(\tau) + f(\tau)\big)d\tau \tag{8}$$

We can get the initial guess $w_0(x)$ by using initial conditions. It is observed that the we are taking better approximation by using fixed value of initial guess in each iteration.

3. Numerical Results

In this section, we present the analytical technique based on the VPM to solve FIDE and VIDE. To show the efficiency of the present method for our problems in comparison with the exact solution by finding the error.

3.1. Problem

Let us suppose the following Fredholm Integro differential equation

$$\frac{du}{dt} = e^t + (e-1) - \int_0^1 u(s)\,ds, \tag{9}$$

with the initial condition $u(0) = 1$,

The exact solution of the above is $u(t) = e^t$. After two iteration the approximate solution of Eq. (9) are shown in Table 1.

Table 1. Numerical results of the problem 1

t	Exact Solution	VPM	Error
0.0	1.	1.	0.000000
0.1	1.105170918	1.168527406	- 0.063356488
0.2	1.221402758	1.331651196	- 0.110248438
0.3	1.349858808	1.491252942	- 0.141394134
0.4	1.491824698	1.649336556	- 0.157511858
0.5	1.648721271	1.808041162	- 0.159319891
0.6	1.822118800	1.969655314	- 0.147536514
0.7	2.013752707	2.136632717	- 0.122880010
0.8	2.225540928	2.311609588	- 0.086068660
0.9	2.459603111	2.497423858	- 0.037820747
1.0	2.718281828	2.697136380	- 0.021145448

*Error = Exact – approximate

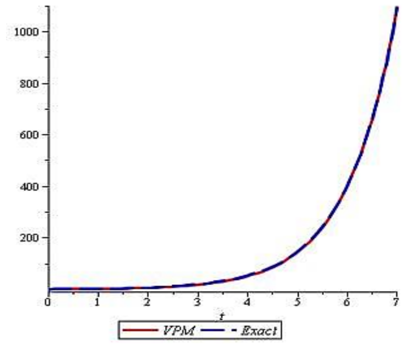

Figure 1. Comparison of VPM and Exact Solution.

3.2. Problem

Suppose the following Fredholm integro-differential equation. (Eq. (10))

$$\frac{du}{dt} = te^t + e^t - t + \frac{1}{2}\int_0^1 su(s)\,dt, \tag{10}$$

with the initial condition $u(0) = 0$.

The exact solution of the above is $u(t) = te^t$. The approximate solution of Eq. (10) are shown in Table 2.

Table 2. Numerical results of the problem 2

t	Exact Solution	VPM	Error
0.0	0.	0.	0.000000
0.1	0.110517091	0.106822728	0.0036943633
0.2	0.244280551	0.229755788	0.0145247634
0.3	0.404957642	0.372919482	0.0320381598
0.4	0.596729879	0.541059019	0.0556708598
0.5	0.824360635	0.739628767	0.0847318684
0.6	1.093271280	0.974887281	0.1183839981
0.7	1.409626895	1.254004438	0.155622457
0.8	1.780432742	1.585182139	0.195250603
0.9	2.213642800	1.977790332	0.235852468
1.0	2.718281828	2.442520116	0.275761712

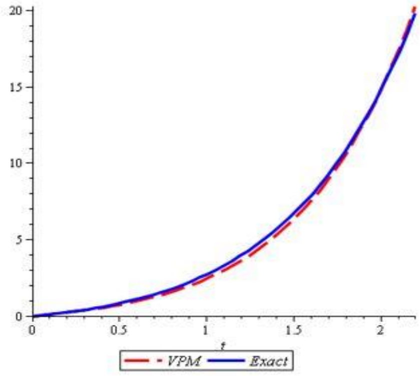

Figure 2. Comparison of VPM and Exact Solution.

3.3. Problem

Suppose the following Fredholm integro-differential equation (Eq. (11))

$$\frac{du}{dt} = 1 - \frac{t}{3} + \int_0^1 tsu(s)\,ds, \qquad (11)$$

with the initial condition $u(0) = 0$,

The exact solution of the above is $u(t) = \mathrm{t}$. The approximate solution of Eq. (11) are shown in Table 3.

Table 3. Numerical results of the problem 3

t	Exact Solution	VPM	Error
0.0	0.	0.	0.000000
0.1	1.0	0.098497916	0.0015020833
0.2	0.2	0.194633333	0.0053666667
0.3	0.3	0.289331250	0.0106687500
0.4	0.4	0.383466666	0.0165333333
0.5	0.5	0.477864583	0.0221354167
0.6	0.6	0.573300000	0.0267000000
0.7	0.7	0.670497916	0.0295020833
0.8	0.8	0.770133333	0.0298666667
0.9	0.9	0.872831250	0.0271687500
1.0	1.0	0.979166666	0.0208333337

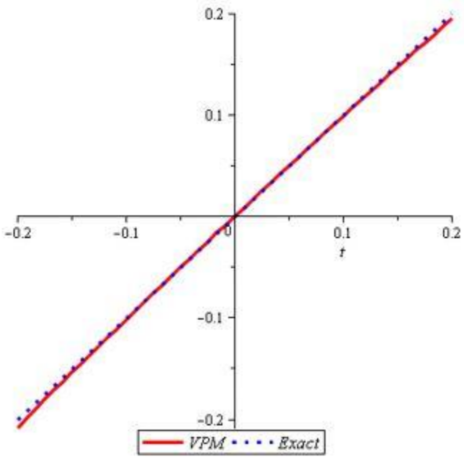

Figure 3. Comparison of VPM and Exact Solution.

3.4. Problem [8]

We consider the Voltera Integro differential equation

$$\frac{du}{dt} = \frac{9}{4} - \frac{5t}{2} - \frac{t^2}{2} - 3e^{-t} - \frac{1}{4}e^{-2t} + \int_0^t (t - s)u^2(s)\,ds, \qquad (12)$$

with the initial condition $u(0) = 2$.

The exact solution of the above is $u(t) = 1 + e^{-t}$. The approximate solution of Eq. (12) are shown in table 4.

Table 4. Numerical results of the problem 4

t	Exact Solution	VPM	Error
0.0	2.	2.0000000	0.0000000
0.1	1.980198673	1.980198700	$-2.7 * 10^{-8}$
0.2	1.960789439	1.960788600	$8.39 * 10^{-7}$
0.3	1.941764534	1.941760400	0.000004134
0.4	1.923116346	1.923103600	0.000012746
0.5	1.904837418	1.904806200	0.000031218
0.6	1.886920437	1.886857100	0.000063337
0.7	1.869358235	1.869242900	0.000115335
0.8	1.852143789	1.851949800	0.000193989
0.9	1.835270211	1.834963900	0.000306311
1.0	1.818730753	1.818270360	0.000460393

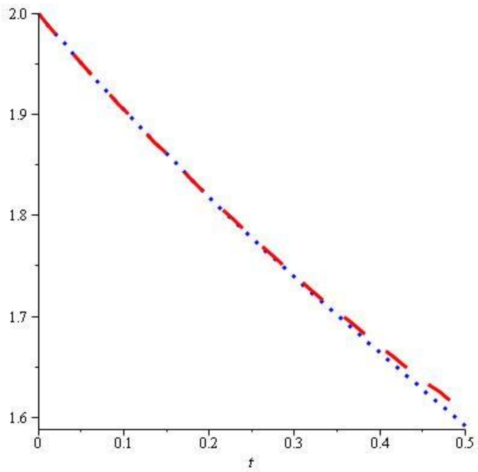

Figure 4. Comparison of VPM and Exact Solution.

3.5. Problem

Lastly we suppose Voltera Integro Differential equation. (Eq. (13)).

$$\frac{du}{dt} = 1 - \frac{t}{2} + \frac{te^{-t^2}}{2} + \int_0^t tse^{-u^2(s)}u(s)\,ds, \qquad (13)$$

with the initial condition $u(0) = 0$.

The exact solution of the above is $u(t) = t$. The approximate solution of Eq. (13) are shown in Table 5.

Table 5. Numerical results of the problem 5

t	Exact Solution	VPM	Error
0.0	0.	0.	0.0000000
0.1	0.1	0.1000000416	- 0.0000000416
0.2	0.2	0.2000026402	- 0.0000026402
0.3	0.3	0.3000297037	- 0.0000297037
0.4	0.4	0.4001640528	- 0.0001640528
0.5	0.5	0.5006123042	- 0.0006123042
0.6	0.6	0.6017809185	- 0.0017809185
0.7	0.7	0.7043559014	- 0.0043559014
0.8	0.8	0.8093768940	- 0.0093768940
0.9	0.9	0.9182979834	- 0.0182979834
1.0	1.0	1.033030140	- 0.033030140

4. Results and Discussion

In above all problems, we are applying successfully our proposed technique and we observe that VPM fastly conveges to the exact solution.

10. Conclusions

Here we are performing VPM to find the analytic solution of linear and nonlinear integro differential equation. We see that after one or two iterations of each problem, VPM gives the best approximate results. Our proposed technique is appropriate for such kind of these linear and nonlinear Integral equations. Our technique is so simple because we get the solution without any perturbation linearization, discretization and Adomian polynomial's.

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